APPROXIMATING PI AND RECURRENCE RELATIONS

Some interesting recurrence relations that converge on pi
Approximating inverse functions using only the first derivative

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Chapter 1

The sine wave

1.1 What is sin(15°)?

This is the sort of question you might see on a YouTube thumbnail while doom-scrolling maths educational videos(shoutouts to blackpenredpen).

At first, it seems like you'll have to pull out some special triangle, and indeed you can do it that way, but some simple algebraic facts can also get the answer:

$sin(2 * 15°) = 0.5 = 2sin(15°)cos(15°)$

$sin^2(15°) + cos^2(15°) = 1$

Solving these equations(and being careful about the minus signs) you can arrive at the answer:

$$sin(15°) = \frac{\sqrt{2 - \sqrt{3}}}{2} \tag{1.1}$$

OK bro that's cool I guess. Time to go to bed...

1.2 Enough sine identities forever

But hold up, the only special about 15°is that it's half of 30°and we knew the value of sin(30°). But now we know the value for 15°, we could find 7.5°, and when we know that, we could find 3.75°and forever more!

In general, the algebra looks like:

$sin(2x) = 2sin(x)cos(x)$

$sin^2(x) + cos^2(x) = 1$

We solve that and get

$$sin(x) = \pm\sqrt{\frac{1 - cos(2x)}{2}} \tag{1.2}$$

Where the $\pm$ depends on if we are on a hump or on a dip.

We can simplify this to

$$sin(x) = \frac{\sqrt{1 + sin(2x)} \pm \sqrt{1 - sin(2x)}}{2} \tag{1.3}$$

Where now the $\pm$ flips in bands of size $\frac{\pi}{2}$ (check this yourself to see what I mean).

What's neat about this is that we have expressed the smaller sine in terms of values of sine that we know. For example:

$$sin(7.5°) = \frac{\sqrt{1 + sin(15°)} - \sqrt{1 - sin(15°)}}{2} \tag{1.4}$$

So now we can make infinitely many YouTube click-bait thumbnails. Nice!

1.3 Going down the well

As we repeat this process more and more, you'll notice that the angles get smaller. Whenever you hear small angles and sine you will immediately think of

$sin(x) \approx x$ if x is small and in radians.

So, now for a key idea: If we apply this rule to the process above, we can express small numbers as extremely messy radicals. Wait, what's the point of that?! Because those small numbers will be in terms of π, and maybe we could then solve for π and get a neat approximation. So let's go for it.

So let's start with a known value of sine: $sin(\frac{\pi}{6}) = \frac{1}{2}$

To make the notation simpler, let's define this as a sequence of reals x_n.

So we have $x_0 = \frac{1}{2}$

Then, from (1.3), we have:

$$x_{n+1} = \frac{\sqrt{1+x_n} - \sqrt{1-x_n}}{2} \tag{1.5}$$

Let's remember that $x_n = sin(\frac{\pi}{3 * 2^{n+1}})$

So, when n is big, we have:

$x_n \approx \frac{\pi}{3 * 2^{n+1}}$

$\Rightarrow 3 * 2^{n+1} * x_n \approx \pi$

$$\Rightarrow lim_{n \to \infty}(3 * 2^{n+1} * x_n) = \pi \tag{1.6}$$

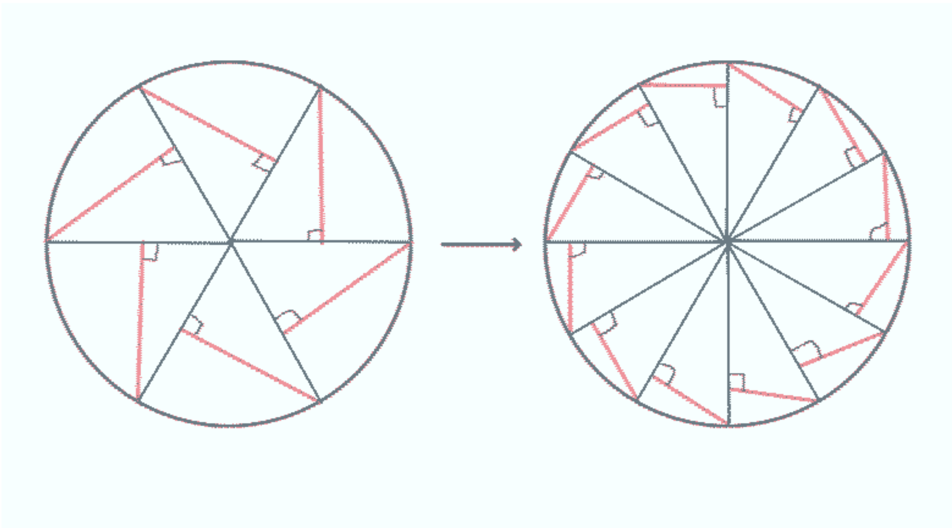
"POLICE!! STOP RIGHT THERE!!!"

Oh no! It's the maths police. They aren't happy because these limits aren't to code.

"HEY!! You can't just re-arrange things like that when you've got $\approx$ signs in the mix!!!"

OK OK I'll prove it officer. Lucky for me, there is a nice geometric way of seeing this. If you think about what $3 * 2^{n+1} * x_n$ means, it's essentially packing $3 * 2^{n+1}$ triangles into half of a circle and adding up the side lengths of the opposite side.

On each step, we double the number of triangles. Here it is shown going from 6 to 12 triangles(note our sequence starts with 12 but I drew 6 for simplicity):



The sides in red are the sines of those triangles, and by multiplying by $3 * 2^{n+1}$ we are adding up all of the little triangles. As we keep on doubling the number of triangles, the total length of the red sides will approach the circumference, which is 2π for this unit circle. So half of that will add up to π.

This is like Archimedes' method, except with jagged edges. And if we imagine trying to rotate each edge by its centre, so that it's normal is facing radially outwards, the edges would not connect. Therefore, we have shown additionally that $3 * 2^{n+1} * x_n < \pi$ for all n. The algebraic way to see this is that $sin(x) < x$ for $\pi > x > 0$. Since x is in radians, it is the arc length, thus our sides are smaller than the arc length they cover. So this is our first theorem:

Theorem 1. If $x_0 = \frac{1}{2}$ and $x_{n+1} = \dfrac{\sqrt{1+x_n} - \sqrt{1-x_n}}{2}$.

$$lim_{n\to\infty}(3 * 2^{n+1} * x_n) = \pi \tag{1.5}$$

and

$$3 * 2^{n+1} * x_n < \pi \tag{1.7}$$

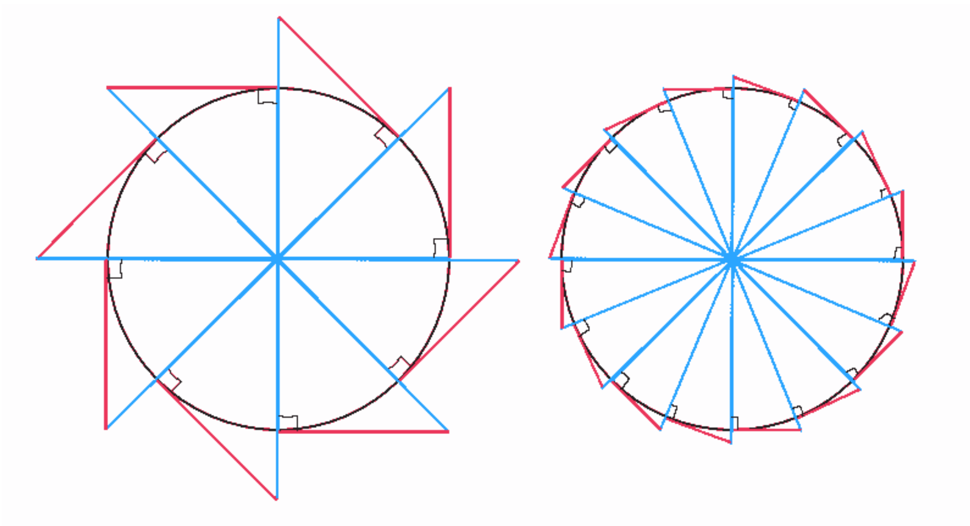
"OK citizen, move along."

Phew! What a close one!

If you were to look at that theorem without hearing all that preamble, you would probably be a bit amazed and puzzled as to how π fits in there. By naming it as x_n we have hidden away the mechanism, but at the same time, the sine function need not exist to define this sequence.

1.4 A tangent

Well you know what other function has a nice small angle approximation? Tangent! So you might think to try this process on that, and indeed it works.



Skipping all the boring algebra and starting with $tan(\frac{\pi}{4}) = 1$, we get:

Theorem 2. If $x_0 = 1$ and $x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n}$.

$$lim_{n\to\infty}(2^{n+2} * x_n) = \pi \tag{1.8}$$

and, since $\frac{\pi}{2} > x > 0 \Rightarrow tan(x) > x$,

$$2^{n+2} * x_n > \pi \tag{1.9}$$

OK and for completeness, what about cos? Well that has a nice half angle formula:

$$cos(x) = \sqrt{\frac{1+cos(2x)}{2}} \tag{1.10}$$

So now we need to be careful since there isn't a nice geometric interpretation of what we are going to do.

Let $x_0 = cos(\frac{\pi}{3}) = 0.5$

Now we get the sequence $x_{n+1} = \sqrt{\frac{1+x_n}{2}}$.

For our small angle approximation we could use $cos(x) \approx 1$ but this will only tell us our sequence tend to 1(which is kind obvious) and won't give us an approximation for π. So we need to use $cos(x) = 1 - \frac{x^2}{2} + \delta(x^3)$, we use a small delta to keep track of the error.

$x_n = cos(\frac{\pi}{3 * 2^n})$

$\Rightarrow x_n = 1 - \frac{\pi^2}{2(3 * 2^n)^2} + \delta(2^{-3n})$

$\Rightarrow \pi^2 = 2(3 * 2^n)^2(1 - x_n) + \delta(2^{-n})$

$\Rightarrow \pi^2 = 3 * 2^n\sqrt{2(1 - x_n) + \delta(2^{-n})}$

So by taking limits as n goes to infinity, we get:

Theorem 3. If $x_0 = \frac{1}{2}$ and $x_{n+1} = \sqrt{\frac{1+x_n}{2}}$.

$$lim_{n\to\infty}(3 * 2^n\sqrt{2(1 - x_n)}) = \pi \tag{1.11}$$

and, since $cos(x) > 1 - \frac{x^2}{2}$(except $x = 0$),

$$3 * 2^n\sqrt{2(1 - x_n)} < \pi \tag{1.12}$$

However, if you examine this theorem then it's really the same as the first.

Chapter 2

Inverse functions

2.1 More trig

Let's think back to Theorem 1. The identity for π dropped out when we started with the known value of $sin(\frac{\pi}{6}) = \frac{1}{2}$ and then, from our iteration process, we were able to remove the sine function and get $/pi$ on it's own. We could try obtaining slightly different versions of this identity by choosing different starting values, but what if we started with some arbitrary value $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$?

Let's say we have: $sin(x) = y$ and let $x_0 = y$ with the same recurrence relation as in Theorem 1. Then we get:

$x_n = sin(\frac{x}{2^n})$

$\Rightarrow x_n = \frac{x}{2^n} + \delta(\frac{x^2}{2^{2n}})$

$$\Rightarrow lim_{n\to\infty}(2^n * x_n) = x \tag{2.1}$$

Remember that x_n is a sequence based solely on y so we have found a way to invert sine using this recurrence relationship.

Why doesn't this work when x is outside of the range $(\frac{-\pi}{2}, \frac{\pi}{2})$? Because the recurrence relation for x_n is from the half angle formula for sine. This relation only holds in the range $(\frac{-\pi}{2}, \frac{\pi}{2})$ unless we flip the sign. Since we are keeping the sign fixed, then the statement $x_n = sin(\frac{x}{2^n})$ only holds with $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$.

This gives us a generalisation of Theorem 1:

Theorem 4. If $f(x) = \frac{\sqrt{1+x} - \sqrt{1-x}}{2}$.

$$lim_{n\to\infty}(2^n f^n(x)) = arcsin(x) \tag{2.2}$$

and, for all non-zero x:

$$|2^n f^n(x)| < |arcsin(x)| \tag{2.3}$$

(Note: $f^n(x)$ means $f(x)$ nested n times.)

We can do the same generalisation to theorem 2:

Theorem 5. If $f(x) = \frac{\sqrt{1+x^2} - 1}{x}$.

$$lim_{n\to\infty}(2^n f^n(x)) = arctan(x) \tag{2.4}$$

and, for all non-zero x:

$$|2^n f^n(x)| > |arctan(x)| \tag{2.5}$$